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The heat conduction problem with temperature-dependent material properties

L. N. TAO

Department of Mechanical and Aerospace Engineering, Illinois Institute of Technology, Chicago, IL 60616, U.S.A.

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Abstract—The heat conduction problem in a semi-infinite region with temperature-dependent material properties is investigated. The problem is linearized by dividing the complete temperature range into a finite number of subintervals. These subintervals are chosen small enough so that the material functions can be considered constant. The resulting problem resembles that of a composite material, except that the interfaces between elements are unknown functions to be solved. It also resembles the Stefan problem of a multi-phase or a polymorphous material. A similarity solution is found and determined. The existence and uniqueness of the solution is thoroughly examined and proved. Finally, when the material functions are temperature independent, the known solution is recovered.

1. INTRODUCTION

In studies of heat conduction problems it is frequently assumed that the material properties are constant and consequently that they are governed by linear heat equations. Because of the linearity, many powerful methods and techniques of linear mathematics can be applied to obtain their solutions. However, in applications where the range of temperature variation is relatively large or where the dependence of the material properties on temperature is appreciable, the problems can no longer be adequately represented by linear heat equations.

There are many known methods for solving nonlinear heat equations. They are usually devised for specific problems [1, 2]. A very useful mathematical method which can be used for both linear and nonlinear problems is the method of similarity transforms. This method has been applied to many heat conduction problems. They have been discussed and summarized in many books [3, 4]. Additional references may be found in refs. [1, 2]. By means of similarity transforms one is able to reduce the governing partial differential equations to ordinary differential equations. The method is, however, only applicable in some specific situations. The domain must be either infinite or semi-infinite and the prescribed boundary and initial conditions must be of some specific forms. In addition, when the material properties are temperature dependent, they are limited to some special expressions, such as a direct power of temperature. Even in these cases, the resulting equations remain nonlinear, it is necessary to solve these equations usually by some approximate or numerical means.

In many physical situations, the material properties expressed in some direct power of temperature, might not be adequate to represent the actual variations. If other expressions, for example, polynomials, are suggested to represent temperature variations, similarity transforms are generally not applicable. It is the purpose of this investigation to offer a new approach which removes this restriction and thus extends the usefulness of similarity transforms to these problems. The material properties, i.e. thermal conductivity, specific heat, and density, can be any arbitrary functions of temperature. This study, for an exploratory purpose, will be restricted to a semi-infinite region with constant initial and boundary conditions.

To overcome the difficulty caused by the temperature-dependent material properties, the complete temperature range is divided into a finite number of subintervals. These subintervals are chosen small enough so that the material properties in each one can be considered to be constant. They then obey the linear heat equations in each subinterval. This formulation resembles the heat conduction in a composite material. However, there are some differences. For a composite material the interfaces between elements are at fixed positions. In the present problem, the interfaces between different elements are the isotherms the positions of which are unknown *a priori*. Because these isotherms change their positions in time, this new problem also resembles another heat conduction problem, the Stefan problem of a multi-phase or a polymorphous material [5, 6]. A major difference between these two problems is that in the present problem the heat flux is continuous across the interface, while in the Stefan problem due to the effect of latent heat the heat flux is discontinuous at the interface.

In the next section the mathematical formulation of the problem is discussed. The temperature range of the problem is divided into a finite number of subintervals. Based on the present formulation, it has a similarity solution. The exact solution is found. Existence and uniqueness of the present solution are then examined in detail. It is proved that the derived similarity solution exists and is unique. Finally, when the temperature range is small and material properties are approximately constant, the solution is known. This known solution can be extracted from the present study.

2. MATHEMATICAL FORMULATION

Consider a semi-infinite material occupying the half space $x > 0$ with an initial constant temperature V and being exposed to another constant temperature U at its surface. Thus one has

$$\rho(T)C(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right], \quad 0 < x < \infty \quad (1)$$

subject to the boundary and initial conditions

$$T(0,t) = U, \quad T(x,0) = V. \quad (2)$$

In this investigation all material properties—ρ (density), C (specific heat) and k (thermal conductivity)—are functions of temperature. Without any loss of generality, one can consider that this is a cooling problem, i.e. $U < V$. As discussed in the Introduction, the problem is first linearized. To accomplish this, the temperature range (U, V) of the material is divided into n arbitrary intervals and they are denoted in ascending order by $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$, with $\theta_0 = U$ and $\theta_n = V$. These temperature intervals need not be of equal magnitude. The basic assumption of this study is now stated: each subinterval is small enough so that all material functions in the subinterval can be treated as constant. This means that the above temperature intervals are so chosen that all variations of the material properties in these intervals are within some preset limits. A small preset limit will of course improve the accuracy, but it will also lengthen the solution.

The temperature in the interval (θ_{i-1}, θ_i) is denoted by T_i and the material constants by ρ_i, C_i and k_i. Introducing the thermal diffusivity

$$\alpha_i = k_i/\rho_i C_i \quad (3)$$

one then has n heat equations $(i = 1, 2, \ldots, n)$

$$\partial T_i/\partial t = \alpha_i(\partial^2 T_i/\partial y^2), \quad s_{i-1} < y < s_i \quad (4)$$

subject to the boundary and initial conditions

$$T_1(0,t) = U, \quad T_n(y,0) = V \quad (5)$$

and interface conditions $(i = 1, 2, \ldots, n-1)$

$$T_i(s_i,t) = T_{i+1}(s_i,t) = \theta_i$$
$$k_i(\partial T_i/\partial y)_{s_i} = k_{i+1}(\partial T_{i+1}/\partial y)_{s_i} \quad (6)$$

where $s_0 = 0$, $s_n = \infty$ and $s_i(t)$ is the position of the isotherm θ_i. It may be noted that s_0 and s_n are the boundaries of the original problem. They are kept at constant temperatures U and V, respectively.

This set of equations resembles the heat conduction in a composite material. But in such a situation the interfaces are at the fixed and prescribed positions, while in the present problem the interfaces are the isotherms which are the unknowns to be solved. Because of the unknown positions of these isotherms, the present formulation also resembles a different heat conduction problem, the Stefan problem of a multiphase or a polymorphous material [7, 8]. A major difference is that there is no latent heat in the present study.

3. MATHEMATICAL SOLUTIONS

The problem now is to find n temperatures T_i and $n-1$ interfaces s_i. A complete similarity solution of this problem is found to exist. Let the solutions of temperatures be

$$T_i(y,t) = a_i + b_i \operatorname{erfc} \xi_i \quad (7)$$

where $\xi_i = y/\sqrt{(4\alpha_i t)}$ and $\operatorname{erfc} z$ is the complementary error function. To complete the solutions, one needs to find $n-1$ interfaces $s_i(t)$, together with the coefficients a_i and b_i, to satisfy the continuity requirements at $n-1$ interfaces. To this end, one introduces

$$s_i(t) = \lambda_i\sqrt{(4\alpha_i t)}, \quad i = 1, 2, \ldots, n-1. \quad (8)$$

Since $s_0 = 0$ and $s_n = \infty$, one introduces for convenience that $\lambda_0 = 0$ and $\lambda_n = \infty$.

Substituting these equations into equations (5) and (6), one obtains

$$\begin{aligned} a_1 &= (\theta_1 - U \operatorname{erfc} \lambda_1)/\operatorname{erf} \lambda_1 \\ b_1 &= (U - \theta_1)/\operatorname{erf} \lambda_1 \\ a_i &= [\theta_i \operatorname{erfc}(\omega_{i-1}\lambda_{i-1}) - \theta_{i-1} \operatorname{erfc} \lambda_i]/\Omega(\lambda_i) \\ b_i &= -\Delta\theta_i/\Omega(\lambda_i) \\ a_n &= V \\ b_n &= (\theta_{n-1} - V)/\operatorname{erfc}(\omega_{n-1}\lambda_{n-1}) \end{aligned} \quad (9)$$

and

$$\begin{aligned} \exp(\lambda_1^2)\operatorname{erf}\lambda_1 &= K_1 \exp(\omega_1^2\lambda_1^2)\Omega(\lambda_2) \\ \exp(\lambda_2^2)\Omega(\lambda_2) &= K_2 \exp(\omega_2^2\lambda_2^2)\Omega(\lambda_3) \\ \exp(\lambda_i^2)\Omega(\lambda_i) &= K_i \exp(\omega_i^2\lambda_i^2)\Omega(\lambda_{i+1}) \\ \exp(\lambda_{n-1}^2)\Omega(\lambda_{n-1}) &= K_{n-1} \exp(\omega_{n-1}^2\lambda_{n-1}^2) \\ &\quad \times \operatorname{erfc}(\omega_{n-1}\lambda_{n-1}) \end{aligned} \quad (10)$$

where

$$\begin{aligned} \omega_i &= (\alpha_i/\alpha_{i+1})^{1/2} \\ K_i &= (k_i/\omega_i k_{i+1})(\Delta\theta_i/\Delta\theta_{i+1}) \\ \Delta\theta_i &= \theta_i - \theta_{i-1} \\ \Omega(\lambda_i) &= \operatorname{erfc}(\omega_{i-1}\lambda_{i-1}) - \operatorname{erfc}\lambda_i. \end{aligned} \quad (11)$$

One has thus obtained the complete solution of the problem.

4. EXISTENCE AND UNIQUENESS

In the preceding section the formal solutions of the problem have been obtained. It remains to show that all coefficients a_i, b_i and λ_i exist and are uniquely determined. Clearly it is seen from equation (9) that if all λ_i's are known, then a_i and b_i are uniquely determined. The proof of the existence of the coefficients λ_i is somewhat involved, because they are governed by a set of interrelated transcendental equations. It is first remarked that all negative values of λ_i are physically inadmissible and are thus ignored because they correspond to the case where isotherms would be lying outside the material region.

In subsequent discussions, some mathematical properties of error integrals are needed. For positive z, both $\operatorname{erfc}(z)$ and $\exp(z^2)\operatorname{erfc}(z)$ are monotonically decreasing from 1 to 0 and $z\exp(z^2)\operatorname{erfc}(z)$ is monotonically increasing from 0 to $1/\sqrt{\pi}$. The first one follows from its integral representation

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \exp(-\xi^2)\,\mathrm{d}\xi \qquad (12)$$

and the latter two follow from the fact [7, 8] that

$$\exp(z^2)\operatorname{erfc}(z)|_0 = 1$$

$$\exp(z^2)\operatorname{erfc}(z)|_\infty = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}z}[\exp(z^2)\operatorname{erfc}(z)] = -2\exp(z^2)\operatorname{ierfc}(z) < 0 \qquad (13)$$

and

$$z\exp(z^2)\operatorname{erfc}(z)|_0 = 0$$

$$z\exp(z^2)\operatorname{erfc}(z)|_\infty = \frac{1}{\sqrt{\pi}}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}[z\exp(z^2)\operatorname{erfc}(z)] = 4\exp(z^2)\operatorname{ierfc}(z) > 0 \qquad (14)$$

where $\mathrm{i}^n\operatorname{erfc}(z)$ is the repeated error integral.

The properties of $\Omega(\lambda_i)$ are now studied. Since $K_{n-1} > 0$, the right-hand side of equation $(10)_4$ is non-negative. This implies that

$$\Omega(\lambda_{n-1}) \geqslant 0 \quad \text{and} \quad \lambda_{n-1} \geqslant \omega_{n-2}\lambda_{n-2}. \qquad (15)$$

From equations $(10)_2$ and $(10)_3$ one finds sequentially that all

$$\Omega(\lambda_i) \geqslant 0 \quad \text{and} \quad \lambda_i \geqslant \omega_i\lambda_i. \qquad (16)$$

In these expressions the equality part is impossible. If the equalities were true, then one has from equation $(10)_4$ that $\lambda_{n-1} = \lambda_{n-2} = \infty$, when $\Omega(\lambda_{n-1}) = 0$. Also from equations $(10)_2$ and $(10)_3$ one obtains that all λ_i's approach ∞. But $\lambda_1 = \lambda_2 = \infty$ could not satisfy equation $(10)_1$, since the value of the left-hand side is infinite and that of the right-hand side is zero. Thus

$$\Omega(\lambda_i) > 0. \qquad (17)$$

In obtaining the last inequality with $i = 1$, one has used

$$\exp(\omega_1^2\lambda_1^2)\operatorname{erfc}(\lambda_2)|_{\lambda_1=\lambda_2=\infty} = \exp(\lambda_2^2)\operatorname{erfc}(\lambda_2)|_\infty = 0. \qquad (18)$$

One may now proceed to the proof of existence and uniqueness of λ_i's. Equations $(10)_1$ and $(10)_2$ can be rewritten as

$$\exp(\omega_1^2\lambda_1^2)\operatorname{erfc}(\lambda_2) = \exp(\omega_1^2\lambda_1^2)\operatorname{erfc}(\omega_1\lambda_1) - \exp(\lambda_1^2)\operatorname{erf}\lambda_1/K_1$$

$$\exp(\lambda^2)[\operatorname{erfc}(\omega_1\lambda_1) - \operatorname{erfc}(\lambda_2)] = K_2\exp(\omega_2^2\lambda_2^2)\Omega(\lambda_3). \qquad (19)$$

A study of these two equations reveals that (i) λ_1 in the first equation is a strictly increasing function of λ_2 from 0 to λ_1^* (a constant), and (ii) λ_2 in the second equation is a strictly increasing function of λ_1 from λ_2^* (a constant) to ∞.

These two assertions are now proved. Consider first equation $(19)_1$. The right-hand side is a strictly decreasing function of λ_1, since $K_1 > 0$. Differentiation with respect to λ_1 yields

$$\exp(\omega_1^2\lambda_1^2)[2\omega_1^2\lambda_1\operatorname{erfc}(\lambda_2) - (2/\sqrt{\pi})(\mathrm{d}\lambda_2/\mathrm{d}\lambda_1)] = \mathrm{d}[\text{R.H.S.}]/\mathrm{d}\lambda_1 < 0. \qquad (20)$$

Therefore

$$\mathrm{d}\lambda_2/\mathrm{d}\lambda_1 > 0. \qquad (21)$$

To find the values at their end points, one can note that when λ_1 vanishes, λ_2 also vanishes. On the other hand, when λ_2 approaches ∞, the left-hand side of equation $(19)_1$ vanishes. The equation reduces to

$$\exp(\omega_1^2\lambda_1^2)\operatorname{erfc}(\omega_1\lambda_1) = \exp(\lambda_1^2)\operatorname{erf}(\lambda_1)/K_1. \qquad (22)$$

This equation has a unique solution, since the left-hand side is strictly decreasing from 1 to 0 and the right-hand side is strictly increasing from 0 to ∞. Thus a unique solution of λ_1 (say λ_1^*) exists at their intersection. Putting these statements together, one can conclude that λ_1 is a strictly increasing function from 0 to λ_1^* when λ_2 varies from 0 to ∞.

To study the second equation, one first observes from equation $(10)_4$ that

$$\exp(\lambda_{n-1}^2)\Omega(\lambda_{n-1}) \qquad (23)$$

is a strictly decreasing function of λ_{n-1}. Thus

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_{n-1}}[\exp(\lambda_{n-1}^2)\Omega(\lambda_{n-1})] < 0. \qquad (24)$$

This yields

$$\exp(-\omega_{n-2}^2\lambda_{n-2}^2)\frac{\mathrm{d}(\omega_{n-2}\lambda_{n-2})}{\mathrm{d}\lambda_{n-1}} > \exp(-\lambda_{n-1}^2) + \sqrt{\pi}\lambda_{n-1}\Omega(\lambda_{n-1}) > 0. \qquad (25)$$

Then by differentiation one has

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_{n-1}}[\exp(-\omega_{n-2}^2\lambda_{n-2}^2)\Omega(\lambda_{n-1})]$$

$$< \frac{2}{\sqrt{\pi}}\exp(-\omega_{n-2}^2\lambda_{n-2}^2)\Omega(\lambda_{n-1})\left[\frac{\Lambda(\omega_{n-2}\lambda_{n-2})}{\mathrm{erfc}(\omega_{n-2}\lambda_{n-2})}\right]$$

$$\times[1-\Lambda(\lambda_{n-1})\Omega(\lambda_{n-1})]-\left[\frac{\Lambda(\lambda_{n-1})}{\mathrm{erfc}(\lambda_{n-1})}\right]$$

$$\times[1-\Lambda(\omega_{n-2}\lambda_{n-2})\Omega(\lambda_{n-1})] \quad (26)$$

where $\Lambda(z)=\sqrt{\pi}\exp(z^2)\,\mathrm{erfc}(z)$ is a monotonic increasing function of z from 0 to 1. Since

$$\mathrm{erfc}(\omega_{n-2}\lambda_{n-2}) > \mathrm{erfc}(\lambda_{n-1})$$

$$\Lambda(\lambda_{n-1}) > \Lambda(\omega_{n-2}\lambda_{n-2})$$

$$\mathrm{d}\lambda_{n-1}/\mathrm{d}\lambda_{n-2} > 0 \quad (27)$$

one finds that

$$\exp(\omega_{n-2}^2\lambda_{n-2}^2)\Omega(\lambda_{n-1}) \quad (28)$$

is a strictly decreasing function of λ_{n-2}. It then follows from equation $(10)_3$ that

$$\exp(\lambda_{n-2}^2)\Omega(\lambda_{n-2}) \quad (29)$$

is also a strictly decreasing function of λ_{n-2}. Repeatedly from equations $(10)_3$ and $(10)_2$ one then obtains that

$$\exp(\omega_2^2\lambda_2^2)\Omega(\lambda_3) \quad (30)$$

is a strictly decreasing function of λ_2. Equation $(19)_2$ can be rewritten as

$$\exp(\lambda_2^2)\,\mathrm{erfc}(\omega_1\lambda_1) = \exp(\lambda_2^2)\,\mathrm{erfc}\,\lambda_2 + K_2\exp(\omega_2^2\lambda_2^2)\Omega(\lambda_3). \quad (31)$$

It is seen that the right-hand side is a strictly decreasing function of λ_2. Differentiation with respect to λ_2 yields

$$\exp(\lambda_2^2)[2\lambda_2\,\mathrm{erfc}(\omega_1\lambda_1)-(2/\sqrt{\pi})(\mathrm{d}\lambda_1/\mathrm{d}\lambda_2)] = \mathrm{d}[\mathrm{R.H.S.}]/\mathrm{d}\lambda_2 < 0. \quad (32)$$

Therefore

$$\mathrm{d}\lambda_1/\mathrm{d}\lambda_2 > 0. \quad (33)$$

To find the values at their end points, one observes that when $\Omega(\lambda_2)=0$ or λ_2 and λ_3 approach ∞, λ_1 also approaches ∞. Additionally, when $\lambda_1=0$, equation $(19)_2$ becomes

$$\exp(\lambda_2^2)\,\mathrm{erf}\,\lambda_2 = K_2\exp(\omega_2^2\lambda_2^2)\Omega(\lambda_3). \quad (34)$$

This equation has a unique solution, say λ_2^*, since the left-hand side is strictly increasing and the right-hand side is strictly decreasing. With these one can conclude that λ_1 is a strictly increasing function of λ_2 from λ_2^* to ∞.

It has been shown that equation $(19)_1$ is a monotonic curve from $(0,0)$ to (λ_2^*,∞) in the (λ_1,λ_2) plane, and that equation $(19)_2$ is a monotonic curve from $(0,\lambda_2^*)$ to (∞,∞). Thus a unique pair (λ_1,λ_2) exists at the intersection of these two curves. This in turn, from equations (10), yields that all λ_i's exist and are unique. Also, from $\Omega(\lambda_i)>0$, one can conclude that

$$0 < s_1 < s_2 < \cdots < s_{n-1} < \infty. \quad (35)$$

These inequalities indicate that in the cooling problem the isotherms of high temperatures are at the right of those of low temperatures. In the heating problem, the effect is reversed. The result agrees with one's expectation. This completes the proof of existence and uniqueness.

5. TEMPERATURE-INDEPENDENT MATERIAL PROPERTIES

If the medium has constant material properties, the solution is greatly simplified. This solution can be readily obtained from the present study. In this case there is only one set of material properties. The interfaces between different elements introduced previously are no longer necessary. By coalescing all interfaces $s_1, s_2, \ldots, s_{n-1}$ into the interface of s_n which is at infinity, and by setting all isotherms of these interfaces equal to V, one has for $i=1,2,\ldots,n-1$

$$s_i=\infty,\quad \theta_i=V,\quad \lambda_i=\infty.$$

Using

$$k_1=k_2=\ldots=k_n=k$$

$$\alpha_1=\alpha_2=\ldots=\alpha_n=\alpha$$

one then eliminates all intermediate elements. There is only one temperature function which is given by

$$T=V+(U-V)\,\mathrm{erfc}\,\xi = U+(V-U)\,\mathrm{erf}\,\xi.$$

This is the well-known solution [4].

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LE PROBLEME DE CONDUCTION THERMIQUE AVEC DES PROPRIETES DU MATERIAU DEPENDANT DE LA TEMPERATURE

Résumé—On étudie le problème de la conduction thermique dans un milieu semi-infini avec des propriétés dépendant de la température. Le problème est linéarisé en divisant le domaine complet de température en un nombre fini d'intervalles. Ceux-ci sont choisis suffisamment petits pour que les fonctions matérielles puissent être considérées comme constantes. Le problème résultant ressemble à celui d'un matériau composite, sauf que les interfaces entre les éléments sont des fonctions inconnues à résoudre. Cela ressemble aussi au problème de Stefan pour un matériau multiphasique ou polymorphe. Une solution en similitude est trouvée et déterminée. L'existence et l'unicité de la solution sont soigneusement examinées et prouvées. Finalement, lorsque les fonctions matérielles sont indépendantes de la température, on retrouve la solution connue.

DAS WÄRMELEITUNGSPROBLEM BEI BERÜCKSICHTIGUNG TEMPERATURABHÄNGIGER STOFFWERTE

Zusammenfassung—Es wird der Wärmeleitvorgang in einem halbunendlichen Körper unter Berücksichtigung temperaturabhängiger Stoffwerte untersucht. Der Vorgang wird linearisiert durch Unterteilen des gesamten Temperaturbereiches in eine endliche Anzahl von Teilintervallen. Die Teilintervalle werden so klein gewählt, daß mit konstanten Stoffwerten gerechnet werden kann. Das resultierende Problem ähnelt dem eines zusammengesetzten Körpers, wobei die Temperatur an den Berührungsflächen benachbarter Elemente bestimmt werden muß. Es existieren ebenso Übereinstimmungen mit dem Stefan-Problem eines mehrphasigen oder polymorphen Stoffes. Eine Ähnlichkeitslösung wird ermittelt. Die Existenz und Eindeutigkeit der Lösung wird eingehend untersucht. Schließlich ergibt sich für den Fall temperaturunabhängiger Stoffeigenschaften die bekannte Lösung.

ЗАДАЧА ТЕПЛОПРОВОДНОСТИ ДЛЯ ТЕЛА, СВОЙСТВА КОТОРОГО ЗАВИСЯТ ОТ ТЕМПЕРАТУРЫ

Аннотация—Исследуется задача теплопроводности в полуограниченной области с учетом зависимости свойств вещества от температуры. Задача линеаризуется путем разделения всего интервала температур на конечное число подинтервалов, которые выбираются достаточно малыми для того, чтобы свойства вещества считались постоянными. Получаемая в результате задача сходна с задачей для композитного материала, за исключением того, что границы между элементами являются неизвестными функциями, подлежащими определению. Она также напоминает задачу Стефана для многофазного или полиморфного материала. Найдено ее автомодельное решение. Доказаны существование и единственность решения. При отсутствии зависимости свойств вещества от температуры получается известное решение.